Math 210A Lecture 2 Notes

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1 Morphisms, Functors, and Commutative Diagrams

1.1 Types of morphisms

Definition 1.1. Let $\mathcal{C}$ be a category. $\mathcal{C}$ is **locally small** if $\mathrm{Hom}(A, B)$ is always a set. $\mathcal{C}$ is **small** if $\mathrm{Obj}(\mathcal{C})$ are a set and $\mathcal{C}$ is locally small.

Definition 1.2. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. f is a **monomorphism** if for any $g, h : U \to X$, $fg = fh$ implies that $g = h$ (f is left-cancellative). f is a **epimorphism** if for any $g, h : Y \to Z$, $gf = hf$ implies that $g = h$ (f is right-cancellative).

Example 1.1. The inclusion $i : \mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the category of rings.

Definition 1.3. If $C \in \mathrm{Obj}(\mathbb{C})$, a **subobject** (A, i) of C is a pair such that $i : A \to C$ is a monomorphism. A **quotient** (B, π) of C is a pair such that $\pi : C \to B$ is an epimorphism.

1.2 Functors

Definition 1.4. Let $\mathcal{C}, \mathcal{D}$ be categories. A **(covariant) functor** $F : \mathcal{C} \to \mathcal{D}$ is a map $F : \mathrm{Obj}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{D})$ and a map $F : \mathrm{Hom}_{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ such that $F(gf) = F(g) \circ F(f)$ and $F(1_X) = 1_{F(X)}$.

There is a dual notion, in which the functor switches the direction of the arrows (composition goes backwards).

Definition 1.5. Let $\mathcal{C}$ be a category. The **opposite category** $\mathcal{C}^{op}$ is the category with the same objects but the morphisms are reversed in direction; i.e. $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ corresponds to $f^{op} \in \mathrm{Hom}_{\mathcal{C}^{op}}(B, A)$.

With this definition, the dual type of functor can be viewed as follows.

Definition 1.6. A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$.

Example 1.2. Forgetful functors are functors which "forget information." The forgetful functor from Ab $\to$ Set takes an abelian group and gives back the underlying set. The forgetful functor from Ring $\to$ Set takes a ring and gives back the underlying set. The forgetful functor from Ring $\to$ Ab takes a ring and gives back the underlying abelian group.

Example 1.3. If $A \in \text{Obj}(\mathcal{C})$, the functor $h_A : C^{op} \to \text{Set}$ is given by $h_A(B) = \text{Hom}_{\mathcal{C}}(B, A)$.

Remark 1.1. A contravariant functor $\mathcal{C} \to \text{Set}$ is sometimes called a **presheaf**.

Definition 1.7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. F is **faithful** if $F : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ is injective for all X, Y. $\mathcal{F}$ is **full** if $F : \text{Hom}_{\mathcal{C}}(X, Y) \to \text{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective for all X, Y. F is **fully faithful** if F is both faithful and full.

Note that a category $\mathcal{E}$ is a subcategory of $\mathcal{C}$ if $\text{Obj}(\mathcal{E}) \subseteq \text{Obj}(\mathcal{C})$ and the inclusion functor $i : \mathcal{E} \to \mathcal{C}$ is full.

Example 1.4. Ab is a full subcategory of Grp.

1.3 Diagrams

Definition 1.8. A **directed graph** G is a set V_G of vertices (dots) and a set E_G of arrows (ordered pairs $(v, w) \in V_G \times V_G$).

Definition 1.9. $\mathbb{F}(G)$ is the **free category** on G if $\text{Obj}(\pi(G)) = V_G$ and $\text{Hom}_{\mathbb{F}(G)}(v, w) = \{e_n e_{n-1} \cdots e_1 : e_i \in E_G(v_{i-1}, v_i), v_0 = v, v_n = w\}$. Composition is concatenation of words.

Definition 1.10. A G**-shaped diagram** in a category $\mathcal{C}$ is a functor $\mathbb{F}(G) \to \mathcal{C}$.

Definition 1.11. A **commutative diagram** is a G-shaped diagram that is constant on $\text{Hom}_{\mathcal{C}}(X, Y)$ for each pair X, Y. In other words, taking any path in the diagram should give the same result. For example, in the diagram below, $g_2 \circ f_1 = f_2 \circ g_1$.

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} \\ C & \xrightarrow{f_2} & D \end{array}$$

Definition 1.12. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\eta : R \to G$ is a collection of maps $\eta_X : F(X) \to G(X)$ for each $X \in \text{obj}(\mathbb{C})$ such that if $f : X \to Y$, then

$$\begin{array}{ccc} F(X) & \xrightarrow{\eta_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) \end{array}$$

Example 1.5. Look at the category Vec_K. Let $V^* = \mathrm{Hom}_K(V, K)$, and let $(-)^* : \mathrm{Vec}_K \to \mathrm{Vec}_K$. There is a natural transformation $\eta : \mathbb{1} \to (-)^{**}$ sending $V \to V^{**}$ by sending $v \mapsto (\lambda \mapsto \lambda(v))$.

Definition 1.13. η is a **natural isomorphism** if each η_X is an isomorphism.

Remark 1.2. In this case, $\{\eta_X^{-1}\}$ will also be a natural transformation.

Definition 1.14. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. F is an **equivalence of categories** if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $FG \to \mathbb{1}_{\mathcal{D}}$, $GF \cong \mathcal{C}$. In this case, G is called a **quasi-inverse**.

Definition 1.15. Let $\mathcal{C}, \mathcal{D}$ be categories. The **functor category** $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category with objects functors $\mathcal{C} \to \mathcal{D}$ and morphisms natural transformations.

Example 1.6. If $\mathcal{C}$ is small and $\mathcal{D}$ is locally small, then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is locally small.

1.4 Yoneda Embedding

Lemma 1.1. *Let $\mathcal{C}$ be a small category. Let* $\mathrm{Yo} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ *be the functor with* $A \mapsto h_A(B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$. *Then* Yo *is a fully faithful functor.*

Proof. To show that Yo is faithful, suppose that $\mathrm{Yo}(f) = \mathrm{Yo}(g)$. Then $f = \mathrm{Yo}(f)_A(1_A) = \mathrm{Yo}(g)_A(1_A) = g$.

We will show that Yo is full next time. □